Methods for Analyzing Resistances of Directed Graphs

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Abstract

Resistance between two nodes is a topic that appears in several areas of science, such as physics, computer science and math. Here, we studied two matrix approach to finding resistances between various vertices of directed graphs, with an emphasis on graphs that are balanced. One approach is using the Moore-Penrose inverse of the matrix, which enables us to take a pseudo-inverse of a matrix that in actuality has no inverse due to it being the Laplacian of a graph. The other approach ties in with ideas from physics such as circuits that are in parallel or series, and builds a way of calculating resistances that is consistent with those methods. We also examine why various requirements are needed, and where conclusions that were drawn are no longer satisfied by dropping the assumptions. We finish up by discussing how this work could be expanded on, and how it can be connected to aspects of biology.

Background

Our main goal for this analysis is to find a way of associating resistances to nodes in a directed graph. Creating some kind of methods to assign numbers to various graphs that have few nodes is relatively simple, but oftentimes these rudimentary systems don't scale up well, either due to cases where the preexisting method doesn't know how to handle the new setup, or the methods are computationally intense, such as by requiring the entire setup to be redone if one small change is made.

Occasionally, we'll make references to whether a particular formula is a metric or not. A metric is a formula $d(x,y)$ that inputs nodes x and y, outputs a distance (here, distances will be interpreted as resistances), such that all 4 of the following properties hold:

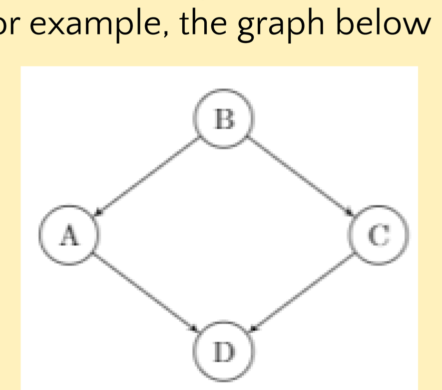
- $d(x,y) \geq 0$
- $d(x,y) = 0$ if and only if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) + d(y,z) \leq d(x,z)$ (the triangle inequality)

A graph is comprised of edges, with each edge connecting two vertices. Directed graphs add directions to each edge, so that they start at one vertex and end at another. Balanced directed graphs are graphs in which for all vertices, the number of edges ending at the vertex equals the number of edges starting at the vertex.

Both of the methods rely on the Laplacian matrix of a graph. This matrix is defined by:

- L_{ii} = Number of edges that start at vertex i
- L_{ij} $(i \neq j)$ = The negative of the number of edges that start at vertex i and end at vertex j

For example, the graph below has the Laplacian shown below



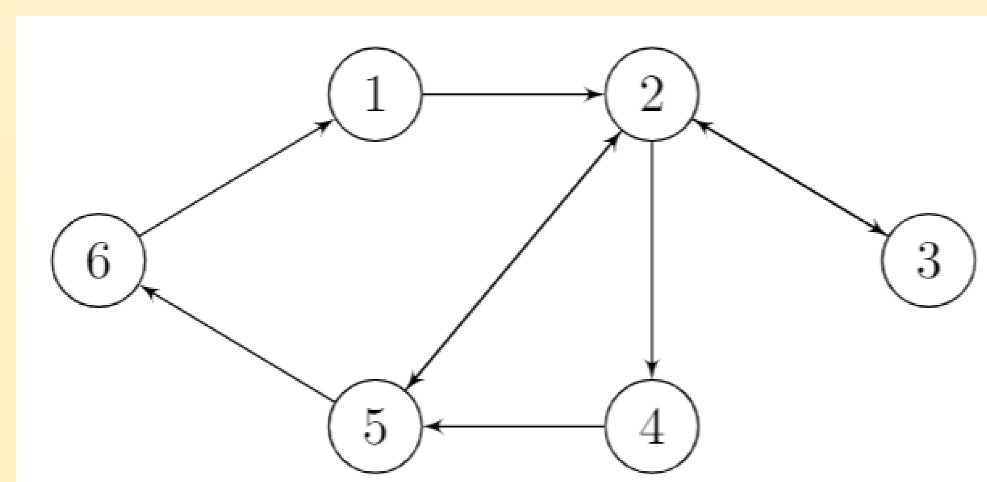
$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With how the Laplacian is defined, for each row, the elements add up to 0, which means that there is no inverse. Many of the theorems involving square matrices assume that the inverse does exist, which we can't make use of here.

Discussion of Moore-Penrose Method

As we've seen, Laplacian matrices don't have inverses. However, there is a pseudo-inverse called the Moore-Penrose inverse for matrices that don't have an inverse, with the Moore-Penrose inverse being "close" to the inverse if it were to exist. Many of the properties that hold for inverses hold for Moore-Penrose inverses as well. For example, for invertible matrices A, $AA^{-1} = I$, the identity matrix. As an immediate result, $AA^{-1}A = A$. This property still transfers over when dealing with Moore-Penrose inverses of matrices, so that $AA^{\dagger}A = A$, where $A^{\dagger}$ denotes the Moore-Penrose inverse of A.

To illustrate how this method works, let us consider the following graph, which is directed and balanced:



The graph above has an associated Laplacian matrix of:

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying the Moore-Penrose inverse via singular value decomposition of the matrix, we get:

$$L^{\dagger} = \begin{bmatrix} \frac{5}{9} & \frac{1}{18} & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{2}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & -\frac{1}{9} \\ -\frac{4}{9} & \frac{1}{18} & \frac{8}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{7}{36} & -\frac{7}{36} & -\frac{13}{36} & \frac{23}{36} & \frac{5}{36} & -\frac{1}{36} \\ -\frac{1}{36} & -\frac{1}{36} & -\frac{7}{36} & -\frac{7}{36} & \frac{11}{36} & \frac{5}{36} \\ \frac{7}{18} & -\frac{1}{9} & -\frac{5}{18} & -\frac{5}{18} & -\frac{5}{18} & \frac{5}{9} \end{bmatrix}$$

As we'll examine further later, one way of defining the resistance given the Moore-Penrose inverse, in such a way that some desired outcomes are achieved, is to define the resistance matrix R by $[r_{ij}] = [l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}]$, where r_{ij} is the resistance from i to j. Given our matrix $L^{\dagger}$, this gives us a resistance matrix of:

$$R = \begin{bmatrix} 0 & \frac{2}{3} & \frac{5}{3} & \frac{17}{12} & \frac{13}{12} & \frac{5}{3} \\ \frac{4}{3} & 0 & 1 & \frac{3}{4} & \frac{5}{12} & 1 \\ \frac{7}{3} & 1 & 0 & \frac{7}{4} & \frac{17}{12} & 2 \\ \frac{19}{12} & \frac{5}{4} & \frac{9}{4} & 0 & \frac{2}{3} & \frac{5}{4} \\ \frac{11}{12} & \frac{7}{12} & \frac{19}{12} & \frac{4}{3} & 0 & \frac{7}{12} \\ \frac{1}{3} & 1 & 2 & \frac{7}{4} & \frac{17}{12} & 0 \end{bmatrix}$$

Discussion of Moore-Penrose Method cont.

The Moore-Penrose inverse has some desired goals, many of which tie in with the definitions of a metric. First, in the r_{ij} formula, if we set $i = j$, we get that $r_{ii} = 0$, which is a desired goal of metrics, and also of resistances in general, in that there should be no resistance to go from a node to itself. Also, as the Moore-Penrose inverse of a balanced directed graph is diagonally dominant, we get that all resistances are non-negative.

Due to the graph being directed, the symmetry property of metrics isn't met. Generally speaking, if there is a directed edge from A to B, the resistance from A to B will be less than that of B to A. The triangle inequality is satisfied, but the validity relies on the assumption that the graph is directed.

Finally for the Moore-Penrose method, given two vertices and a directed edge both directions between them, the resistance is 1, as would be expected in this simple setup.

Discussion of Physics Method

Oftentimes the first time people are exposed to the ideas of resistance (and current) is in physics classes, and this method works by ensuring that resistances for systems in parallel or series give the same values as the values typically used in physics.

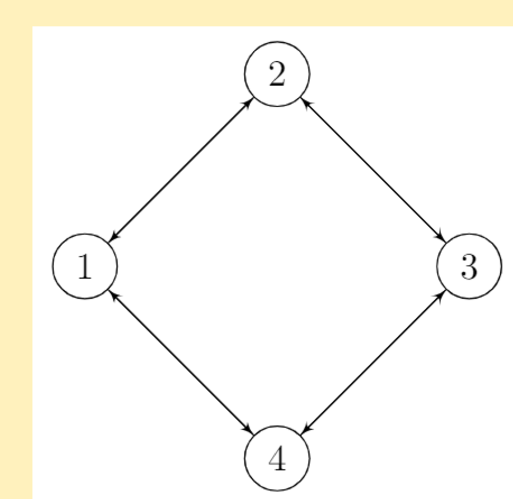
In this method, the formula for the resistance matrix Ω is:

$$\Omega = \nabla|\phi)(\phi| + |\phi)(\phi|\nabla - 2[Q/(\Delta - A)]$$

In the above formula, we have the following variable definitions:

- $\Delta - A$: Laplacian of the graph
- $|\phi)$: Eigenvector of the 0 eigenvalue as a column
- $(\phi|$: Eigenvector of the 0 eigenvalue as a row
- $Q = I - \frac{1}{(\phi|\phi)}|\phi)(\phi|$
- ∇= $(i|\ Q/(\Delta - A)|i)$ on the main diagonal, and 0 elsewhere

We can explore how this formula can be applied to the following graph:



Applying the formula to the above graph, we get:

$$\Omega = \begin{bmatrix} 0 & \frac{3}{4} & 1 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} & 1 \\ 1 & \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & 1 & \frac{3}{4} & 0 \end{bmatrix}$$

If one considers all of the edges to have the same resistance of 1, then the formulas that series/parallel circuits give are equal to the values in the matrix. Furthermore, if all edges are undirected (no directed edge without the equivalent edge in the opposite direction), then the resistance values given satisfy all of requirements of metric distances.

Conclusion

As with most spaces that have metrics associated with them, there are various metrics that can be used for that space. Depending on what one desires from the metric, there are different metrics for resistances of graphs. The Moore-Penrose inverse is frequently used when dealing with matrix operations for matrices that have no inverse, so if one already has the Moore-Penrose inverse already calculated, this approach could result in less additional coding time and storage, which can be useful when dealing with graphs with a large number of vertices.

Using the physics method allows us to calculate resistances circuits with parallel or series circuits, including when there are multiple cycles in the graph, an area where introductory physics formulas tend to either fail at or require solving a recursive relation.

Future Work

There are two main goals for what areas of this area of mathematics I want to explore further: the biological aspect and the mathematical aspect.

For the biological part, graphs are oftentimes used when all elements of the domain can be classified into exactly one of a list of species/states (such as susceptible, infected and removed for disease spread), with the edge weights representing how the nodes interact with each other. One possible area that the resistances in particular can be used for is calculating how dependent other species are on a given species, and whether other species could survive the removal of a given species, or also see their population reduced to zero.

As for the mathematical aspect, a lot of the exploration thus far has been made with several assumptions, such as having a balanced graph, and that all the edges had the same weight (set to 1). From this, the natural question arises of what happens when those assumptions are removed or replaced with other assumptions. This also allows us to expand our biological applications, since when modelling real-life situations, some of the assumptions that we made are unlikely to be valid.

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References:

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