Methods for Analyzing Resistances of Directed Graphs

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Abstract

Resistance between two nodes is a topic that appears in several areas of science, such as physics, computer science and math. Here, we studied two matrix approaches to finding resistances between various vertices of directed graphs, with an emphasis on graphs that are balanced. One approach is using the Moore-Penrose inverse of the matrix, which enables us to take a pseudo-inverse of a matrix that in actuality has no inverse due to it being the Laplacian of a graph. The other approach lies in ideas from physics such as circuits that are in parallel or series, and builds a way of calculating resistances that is consistent with those methods. We also examine why various requirements are needed, and where conclusions that were drawn are no longer satisfied by dropping the assumptions. We finish up by discussing how this work could be expanded on, and how it can be connected to aspects of biology.

Discussion of Moore-Penrose Method

As we’ve seen, Laplacian matrices don’t have inverses. However, there is a pseudo-inverse called the Moore-Penrose inverse for matrices that don’t have an inverse, with the Moore-Penrose inverse being “close” to the inverse if it were to exist. Many of the properties that hold for inverses hold for Moore-Penrose inverses as well. For example, for invertible matrices $A$, $A^+ = A^{-1}$, the identity matrix. As an immediate result, $A^+A = A$. This property still transfers over when dealing with Moore-Penrose inverses of matrices, so that $AA^+A = A$, where $A^+$ denotes the Moore-Penrose inverse of $A$.

To illustrate how this method works, let us consider the following graph, which is directed and balanced:

![Directed Graph](image)

The graph above has an associated Laplacian matrix of:

$$L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Applying the Moore-Penrose inverse via singular value decomposition of the matrix, we get:

$$L^+ = \begin{bmatrix}
5 & 7 & 9 & 1 & 1 \\
7 & 5 & 7 & 1 & 1 \\
9 & 7 & 5 & 1 & 1 \\
1 & 1 & 1 & 5 & 1 \\
1 & 1 & 1 & 1 & 5
\end{bmatrix}$$

Discussion of Physics Method

Oftentimes the first time people are exposed to the ideas of resistance (and current) is in physics classes, and this method works by ensuring that resistances for systems in parallel or series give the same values as the values typically used in physics.

In this method, the formula for the resistance matrix $\Omega$ is:

$$\Omega = \frac{1}{2} \sum_{i} \left( \frac{1}{\epsilon_i} \right)$$

In the above formula, we have the following variable definitions:

- $A$: $A_i$ Laplacian of the graph
- $\phi_i$: Eigenvector of the $0$ eigenvalue as a column
- $\Omega_i$: Eigenvector of the $0$ eigenvalue as a row
- $\Omega = \frac{1}{2} \sum_{i} (\phi_i \phi_i^T)$ on the main diagonal, and $0$ elsewhere

We can explore how this formula can be applied to the following graph:

![Directed Graph](image)

Applying the formula to the above graph, we get:

$$R = \begin{bmatrix}
2 & 0 & 5 & 17 & 13 & 5 \\
0 & 3 & 3 & 13 & 12 & 3 \\
5 & 0 & 1 & 4 & 1 & 3 \\
17 & 3 & 4 & 0 & 1 & 3 \\
13 & 12 & 1 & 0 & 1 & 3 \\
5 & 3 & 12 & 3 & 1 & 0
\end{bmatrix}$$

Discussion of Physics Method cont.

The Moore-Penrose inverse has some desired goals, many of which tie in with the definitions of a metric. First, in the $\gamma_i$ formula, if we set $\gamma_i = 0$, which is a desired goal of metrics, and also of resistances in general, in that there should be no resistance to go from a node to itself. Also, as the Moore-Penrose inverse of a balanced directed graph is diagonally dominant, we get that all resistances are non-negative.

Due to the graph being directed, the symmetry property of matrices isn’t met. Generally speaking, if there is a directed edge from $A$ to $B$, the resistance from $A$ to $B$ will be less than that of $B$ to $A$. The triangle inequality is satisfied, but the validity relies on the assumption that the graph is directed.

For the Moore-Penrose method, given two vertices and a directed edge both directions between them, the resistance is 1, as would be expected in this simple setup.

Conclusion

As with most spaces that have metrics associated with them, there are various metrics that can be used for that space. Depending on what one desires from the metric, there are different metrics for resistances of graphs. The Moore-Penrose inverse is frequently used when dealing with matrix operations for matrices that have no inverse, so if one already has the Moore-Penrose inverse already calculated, this approach could result in less additional coding time and storage, which can be useful when dealing with graphs with a large number of vertices.

Using the physics method allows us to calculate resistances circuits with parallel or series circuits, including when there are multiple cycles in the graph, an area where introductory physics formulas tend to either fail at or require solving a recursive relation.

Future Work

There are two main goals for what areas of mathematics I want to explore further: the biological aspect and the mathematical aspect. For the biological part, graphs are oftentimes used when all elements of the domain can be classified into exactly one of a list of species/states (such as susceptible, infected and removed for disease spread), with the edge weights representing how the nodes interact with each other. One possible area that the resistances in particular can be used for is calculating how dependent other species are on a given species, and whether other species could survive the removal of a given species, or also see their population reduced to zero.

As for the mathematical aspect, a lot of the exploration thus far has been made with several assumptions, such as having a balanced graph, and that all the edges had the same weight (set to 1). From this, the natural question arises of what happens when those assumptions are removed or replaced with other assumptions. This also allows us to expand our biological applications, since when modeling real-life situations, some of the assumptions that we made are unlikely to be valid.

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References: